Exercise 41

Calculate y'.

$$y = \frac{\sqrt{x+1}\,(2-x)^5}{(x+3)^7}$$

Solution

Take the logarithm of both sides to simplify the right side.

$$\begin{aligned}
\ln y &= \ln \frac{\sqrt{x+1}\,(2-x)^5}{(x+3)^7} \\
&= \ln[\sqrt{x+1}\,(2-x)^5] - \ln(x+3)^7 \\
&= \left[\ln\sqrt{x+1} + \ln(2-x)^5\right] - \ln(x+3)^7 \\
&= \frac{1}{2}\ln(x+1) + 5\ln(2-x) - 7\ln(x+3)
\end{aligned}$$

Take the derivative of both sides with respect to x.

$$\begin{aligned}
\frac{d}{dx}(\ln y) &= \frac{d}{dx}\left[\frac{1}{2}\ln(x+1) + 5\ln(2-x) - 7\ln(x+3)\right] \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{1}{2}\left[\frac{d}{dx}\ln(x+1)\right] + 5\left[\frac{d}{dx}\ln(2-x)\right] - 7\left[\frac{d}{dx}\ln(x+3)\right] \\
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{2}\left(\frac{1}{x+1}\right)\cdot\frac{d}{dx}(x+1) + 5\left(\frac{1}{2-x}\right)\cdot\frac{d}{dx}(2-x) - 7\left(\frac{1}{x+3}\right)\cdot\frac{d}{dx}(x+3) \\
&= \frac{1}{2}\left(\frac{1}{x+1}\right)\cdot(1) + 5\left(\frac{1}{2-x}\right)\cdot(-1) - 7\left(\frac{1}{x+3}\right)\cdot(1) \\
&= \frac{1}{2(x+1)} - \frac{5}{2-x} - \frac{7}{x+3} \\
&= \frac{1(2-x)(x+3) - 5[2(x+1)](x+3) - 7[2(x+1)](2-x)}{2(x+1)(2-x)(x+3)} \\
&= \frac{(-x^2-x+6) - (10x^2+40x+30) - (-14x^2+14x+28)}{2(x+1)(2-x)(x+3)} \\
&= \frac{3x^2-55x-52}{2(x+1)(2-x)(x+3)}
\end{aligned}$$

Multiply both sides by y.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{3x^2 - 55x - 52}{2(x+1)(2-x)(x+3)}y \\
&= \frac{3x^2 - 55x - 52}{2(x+1)(2-x)(x+3)}\left[\frac{\sqrt{x+1}\,(2-x)^5}{(x+3)^7}\right] \\
&= \frac{(3x^2 - 55x - 52)(2-x)^4}{2\sqrt{x+1}(x+3)^8}
\end{aligned}$$